ECE2521: Analysis of Stochastic Processes

Homework 2: Topics 3 and 4 (Counting and Random Variables)

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Assigned Problems (11)

Textbook Problems (2.1 - 2.7): 2.43, 2.46, 2.79, 2.81, 2.87, 2.100, and 2.102
Application Problems (2.8 - 2.11): Cards, Parking, Coins, and Continuity

Textbook: (2.1 – 2.7)

2.43. A Web site requires that users create a password with the following specifications:
- Length of 8 to 10 characters
- Includes at least one special character {!, @, #, $, %, ^, &, *, (,), +, =, {, }, |, <, >, \, _, -, [,], /, ?}
- No spaces
- May contain numbers (0–9), lower and upper case letters (a–z, A–Z)
- Is case-sensitive.

How many passwords are there? How long would it take to try all passwords if a password can be tested in 1 microsecond?

2.46. Ordering a "deluxe" pizza means you have four choices from 15 available toppings. How many combinations are possible if toppings can be repeated? If they cannot be repeated? Assume that the order in which the toppings are selected does not matter.

2.79. One of two coins is selected at random and tossed three times. The first coin comes up heads with probability p_1 and the second coin with probability $p_2 = 2/3 > p_1 = 1/3$.
(a) What is the probability that the number of heads is k?
(b) Find the probability that coin 1 was tossed given that k heads were observed, for $k = 0, 1, 2, 3$.
(c) In part b, which coin is more probable when k heads have been observed?
(d) Generalize the solution in part b to the case where the selected coin is tossed m times. In particular, find a threshold value T such that when $k > T$ heads are observed, coin 1 is more probable, and when $k < T$ are observed, coin 2 is more probable.
(e) Suppose that $p_2 = 1$ (that is, coin 2 is two-headed) and $0 < p_1 < 1$. What is the probability that we do not determine with certainty whether the coin is 1 or 2?

2.81. A ternary communication system is shown in Fig. P2.4. Suppose that input symbols 0, 1, and 2 occur with probability 1/3 respectively.
(a) Find the probabilities of the output symbols.
(b) Suppose that a 1 is observed at the output. What is the probability that the input was 0? 1? 2?

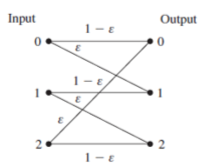


FIGURE P2.4

2.87. Let A, B, and C be events with probabilities $P[A]$, $P[B]$, and $P[C]$.
(a) Find $P[A \cup B]$ if A and B are independent.
(b) Find $P[A \cup B]$ if A and B are mutually exclusive.
(c) Find $P[A \cup B \cup C]$ if A, B, and C are independent.
(d) Find $P[A \cup B \cup C]$ if A, B, and C are pairwise mutually exclusive.

2.100. Each of n terminals broadcasts a message in a given time slot with probability p.
(a) Find the probability that exactly one terminal transmits so the message is received by all terminals without collision.
(b) Find the value of p that maximizes the probability of successful transmission in part a.
(c) Find the asymptotic value of the probability of successful transmission as n becomes large.

2.102. A machine makes errors in a certain operation with probability p. There are two types of errors. The fraction of errors that are type 1 is α, and type 2 is $1 - \alpha$.
(a) What is the probability of k errors in n operations?
(b) What is the probability of k_1 type 1 errors in n operations?
(c) What is the probability of k_2 type 2 errors in n operations?
(d) What is the joint probability of k_1 and k_2 type 1 and 2 errors, respectively, in n operations?

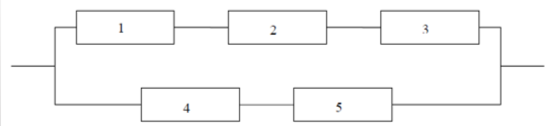
Application: (2.8 – 2.11)

Problem 2.8 : : Draw the top 8 cards from a well-shuffled standard 52-card deck. Find the probability that
(a) The 8 cards include exactly 4 queens.
(b) The 8 cards include exactly 2 kings.
(c) The 8 cards include exactly 4 queens or exactly 2 kings or both.

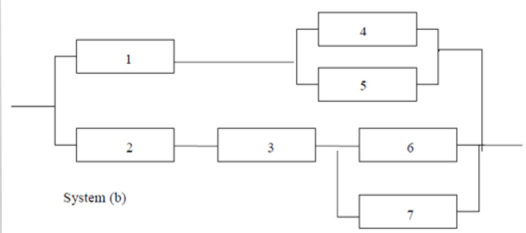
Problem 2.9 :. Twenty distinct cars park in the same parking lot everyday. Ten of these cars are US-made, while the other ten are foreign-made. This parking lot has exactly twenty spaces, and all are in a row. However, the drivers have different schedules on any given day, so the position any car might take on a certain day is random.
(a) In how may different ways can the cars line-up?
(b) What is the probability that on a given day, the cars will park in such a way that they are of alternate makes?

Problem 2.10 : We are given three coins: one has heads in both faces, the second has tails in both faces, and the third has a head in one face and a tail in the other. We choose a coin at random (with equal probabilities), toss it, and the result is heads. What is the probability that the opposite face is tails?

Problem 2.11: Two communication systems (shown below) are composed from several links, where, for proper operation, a connection must be available between two end points of each system. For a link to work properly, it must provide a connection across its end points. The links are assumed to fail independently of each other. Each link is numbered with i, and we assume that the probability that link i fails is equal to q_i. Find the probability of failure of each system, where $q_1 = 0.1$, $q_2 = q_3 = 0.05$, $q_4 = q_5 = 0.15$, $q_6 = q_7 = 0.2$.



System (a)



System (b)

Textbook Problems 1 through 7

2.1 (2.43):

Since format is at least 1 special character anywhere in the password:

special characters (≥ 1): 24 options

total options (fill length): 86 options

valid = total - invalid

Therefore total number of 8-10 character length passwords is:

$$(86^8 - 62^8) + (86^9 - 62^9) + (86^{10} - 62^{10}) = \boxed{2.154 * 10^{19}\ options}$$

If each guess takes 1 microsecond to process then to try all combinations it will take:

$$2.154 * 10^{19}\ options * \frac{1 * 10^{-6}\ seconds}{option} = \boxed{2.154 * 10^{13} seconds \approx 683028.9\ years}$$

2.2 (2.46):

Without Ordering Implies Combination. For 15 choose 4, there are:

$$without\ replacement: \binom{N}{k} = \binom{15}{4} = \boxed{1365\ combinations}$$

$$with\ replacement: \binom{N+k-1}{k} = \binom{18}{4} = \boxed{3060\ combinations}$$

2.3 (2.79):

(a) Using Bernoulli's for 3 flips of a single random coin, defining heads as success:

$$P[k\ heads] = P[Coin1] \cap P[k\ heads|Coin1] \cup P[Coin2] \cap P[kheads|Coin2]$$

$$= \boxed{\frac{1}{2} * (\binom{3}{k} * p_1^k * (1-p_1)^{3-k}) + \frac{1}{2} * (\binom{3}{k} * p_2^k * (1-p_2)^{3-k})} \quad [where: p_1 = \tfrac{1}{3} \text{ and } p_2 = \tfrac{2}{3}]$$

(b) Using Baye's Rule:

$$P[Coin1|k\ heads] = \frac{P[k\ heads|Coin1] * P[Coin1]}{P[k\ heads]}$$

$$= \boxed{\frac{\frac{1}{2} * (\binom{3}{k} * p_1^k * (1-p_1)^{3-k})}{\frac{1}{2} * (\binom{3}{k} * p_1^k * (1-p_1)^{3-k}) + \frac{1}{2} * (\binom{3}{k} * p_2^k * (1-p_2)^{3-k})}}$$

Using MATLAB to evaluate the expression at k = {0, 1, 2, 3}, the probabilities are:

	P[Coin 1]	P[Coin 2]
k = 0	$\frac{8}{9}$	$\frac{1}{9}$
k = 1	$\frac{6}{9}$	$\frac{3}{9}$
k = 2	$\frac{3}{9}$	$\frac{6}{9}$
k = 3	$\frac{1}{9}$	$\frac{8}{9}$

(c) Finding k such that P[Coin 1] > P[Coin 2]:

$$P[Coin1|k\ heads] > P[Coin2|k\ heads] \tag{1}$$
$$\frac{\frac{1}{2}*(\binom{3}{k}*p_1^k*(1-p_1)^{3-k})}{\frac{1}{2}*(\binom{3}{k}*p_1^k*(1-p_1)^{3-k})+...} > \frac{\frac{1}{2}*(\binom{3}{k}*p_2^k*(1-p_2)^{3-k})}{\frac{1}{2}*(\binom{3}{k}*p_1^k*(1-p_1)^{3-k})+...} \tag{2}$$
$$p_1^k*(1-p_1)^{3-k} > p_2^k*(1-p_2)^{3-k} \tag{3}$$
$$(\frac{p_1}{p_2})^k*(\frac{1-p_1}{1-p_2})^{3-k} < 1 \tag{4}$$
$$(\frac{p_1}{p_2})^k*(\frac{1-p_1}{1-p_2})^3*(\frac{1-p_2}{1-p_1})^k < 1 \tag{5}$$
$$(\frac{p_1*(1-p_2)}{p_2*(1-p_1)})^k < (\frac{1-p_2}{1-p_1})^3 \tag{6}$$
$$k < \frac{3*log(\frac{1-p_2}{1-p_1})}{log(\frac{p_1*(1-p_2)}{p_2*(1-p_1)})} \tag{7}$$
$$k < \frac{3*log(\frac{1}{2})}{log(\frac{1}{4})} \tag{8}$$
$$k < \boxed{1.5\ heads} \tag{9}$$

Therefore it is more likely to be Coin 1 when the coin lands k = {0 , 1} heads out of 3 flips. It is more likely to be Coin 2 when there are k = $\{k \mid k > 1, k \in \mathbb{N}\}$ heads out of 3 flips.

(d) For general m flips, from (7) above:

$$\boxed{T = \frac{m*log(\frac{1-p_2}{1-p_1})}{log(\frac{p_1*(1-p_2)}{p_2*(1-p_1)})}}$$

It is more likely to be Coin 1 when T is greater than the expression. It is more likely to be Coin 2 when T is less than the expression.

(e) You can know for certain which coin it is if there are any heads, implying that you cannot know for certain which coin it is if they are all heads, from (b) occuring with probability:

$$P[Coin1 \mid m\ heads] = \frac{\frac{1}{2}*(\binom{m}{m}*p_1^m*(1-p_1)^{m-m})}{\frac{1}{2}*(\binom{m}{m}*p_1^m*(1-p_1)^{m-m})+\frac{1}{2}*(\binom{m}{m}*p_2^m*(1-p_2)^{m-m})}$$
$$= \boxed{\frac{p_1^m}{p_1^m+1}} \qquad , \qquad [note: 0^0 = 1]$$

2.4 (2.81):

(a) Probability of each output is:

$$P[0_{out}] = P[1-\epsilon \mid P[0_{in}]] \ \cap P[\epsilon \mid P[2_{in}]]$$
$$P[1_{out}] = P[1-\epsilon \mid P[1_{in}]] \ \cap P[\epsilon \mid P[0_{in}]]$$
$$P[2_{out}] = P[1-\epsilon \mid P[2_{in}]] \ \cap P[\epsilon \mid P[1_{in}]]$$
$$P[0_{in}] = P[1_{in}] = P[2_{in}] \quad \therefore \quad P[0_{out}] = P[1_{out}] = P[2_{out}] = \boxed{\frac{1}{3}}$$

(b) Probability of inputs if output observed is 1, from figure:

$$P[0_{in} \mid 1_{out}] = \epsilon$$
$$P[1_{in} \mid 1_{out}] = 1 - \epsilon$$
$$P[2_{in} \mid 1_{out}] = 0$$

2.5 (2.87):

Find Events given conditions:

(a) $Independent : P[A \cup B] = P[A] * P[B]$

(b) $Mutually\, exclusive : P[A \cup B] = P[A] + P[B]$

(c) $Independent : P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[A \cap C] + P[A \cap B \cap C]$

(d) $Pairwise\, mutually\, exclusive : P[A \cup B] = P[A] + P[B] + P[C]$

2.6 (2.100):

n terminals broadcast with probability p:

(a) Similar to [2.1]:

$$P[=1broadcasts] = \boxed{p * n * (1-p)^{n-1}}$$

(b) Find p such that P[=1 broadcasts]$_{max}$ at critical point of 1st derivative:

$$\frac{d}{dp}(p * n * (1-p)^{n-1}) = 0 \tag{1}$$
$$(p * n) * -(n-1)(1-p)^{n-2}] + [(1-p)^{n-1} * n] = 0 \tag{2}$$
$$[-p * n * (n-1)(1-p)^{n-2}] + [(1-p)^{n-1} * n] = 0 \tag{3}$$
$$(1-p)^{n-1} = p * (n-1)(1-p)^{n-2} \tag{4}$$
$$(1-p)^{(n-1)-(n-2)} = p * (n-1) \tag{5}$$
$$1 - p = p * (n-1) \tag{6}$$
$$p(n+1-1) = 1 \tag{7}$$
$$p = \boxed{\frac{1}{n}} \tag{8}$$

(c) substituting p in and taking limit as n becomes large:

$$\begin{aligned} P[=1broadcasts] &= \frac{1}{n} * n * (1 - \frac{1}{n})^{n-1} \\ &= (1 - \frac{1}{n})^{n-1} \\ &= \frac{(1-\frac{1}{n})^n}{(1-\frac{1}{n})} \\ lim_{n\to\infty} \frac{(1-\frac{1}{n})^n}{(1-\frac{1}{n})} &= \frac{e^{-1}}{1} \\ &\approx \boxed{0.368} \end{aligned}$$

2.7 (2.102):

$P[error] = p, \quad P[type1|error] = \alpha, \quad and \quad P[type2|error] = 1 - \alpha$:

(a) Using Bernoulli's, k errors in n operations, similar to [2.3]:

$$P[k\ errors] = \binom{n}{k} * p^k * (1-p)^{n-k}$$

(b) Because type 1 and type 2 errors form a partition:

$$P[k_1\ type1s] = \binom{n}{k_1} * (\alpha * p)^{k_1} * (1 - (\alpha * p))^{n-k_1}$$

(c) Similarly:

$$P[k_2\ type2s] = \binom{n}{k_2} * ((1-\alpha) * p)^{k_2} * (1 - ((1-\alpha) * p))^{n-k_2}$$

(d) Joint probability - write (a) in terms of (b) and (c):

$$P[k_1, k_2] = \binom{n}{k_1, k_2} * [(\alpha * p)^{k_1} * ((1-\alpha) * p)^{k_2}] * (1-p))^{n-[k_1+k_2]}$$

Application Problems 8 through 11

2.8 (Cards):

Draw top 8 cards, find probabilities of:

(a)

$$P[=4\,queens] = \frac{\binom{4}{4} * \binom{48}{4}}{\binom{52}{8}} \approx \boxed{0.02586\%}$$

(b)

$$P[=2\,kings] = \frac{\binom{4}{2} * \binom{48}{6}}{\binom{52}{8}} \approx \boxed{9.784\%}$$

(c)

$$P[=4\,queens \cup =2\,kings] = P[=4queens] + P[=2kings] - P[=4queens \cap =2kings]$$

$$P[=4queens \cap =2kings] = \frac{\binom{4}{4} * \binom{4}{2} * \binom{44}{2}}{\binom{52}{8}} \approx 0.00075425\%$$

$$\therefore \quad P[=4\,queens \cup =2\,kings] = 0.02586\% + 9.784\% - 0.0000075425\% \approx \boxed{9.809\%}$$

2.9 (Cars):

(a) For 20 distinct cars taken 20 at a time, the result can be calculated using permutations:

$$P_{20}^{20} = \frac{20!}{(20-20)!} = 2.433e18\,orderings$$

(b) Using a similar method to combinations with restrictions:

$$P[alternating] = \frac{Ways_{Us} * Ways_{Foreign}}{Ways_{Total}}$$

$$= \frac{10!^2 * 2}{20!} \approx \boxed{0.0011\%} \quad \text{(note *2 to account if F-US vs US-F)}$$

2.10 (Coins):

After the given information of landing heads, a new outcome space can be defined:

$$S_{Coins} = \{\, Side_1|Coin_{H-H}\ ,\ Side_2|Coin_{H-H}\ ,\ Side_1|Coin_{H-T} \,\}$$

This can be simplified by only looking at the possible hidden sides:

$$S_{hidden} = \{\, Heads, Heads, Tails \,\}$$

Assuming equal probability of all outcomes, the probability of the other side being tails is: $\boxed{\frac{1}{3}}$

2.11 (Continuity):

(a) For system (a) to fail, find the probability that either serial sides fail (independent individual events):

$$P[fail_{sysA}] = (q_1 \cup q_2 \cup q_3) \cap (q_4 \cup q_5)$$

$$= (0.1 + 0.05 + 0.05 - (0.1 * 0.05 + 0.1 * 0.05 + 0.05 * 0.05) + 0.1 * 0.05 * 0.05)...$$

$$* (0.15 + 0.15 - (0.15 * 0.15) = \boxed{0.0521}$$

(b) For system (b) to fail, find the probability that fail-safes fail (independent individual events):

$$
\begin{aligned}
P[fail_{sysB}] &= (q_1 \cup (q_4 \cap q_5)) \cap (q_2 \cup q_3 \cup (q_6 \cap q_7)) \\
&= (0.1 + (0.15 * 0.15) - (0.1 * (0.15 * 0.15))... \\
&\quad * (0.05 + 0.05 + (0.2 * 0.2)... \\
&\quad - (0.05 * 0.05 + 0.05 * (0.2 * 0.2) + 0.05 * (0.2 * 0.2)) + 0.05 * 0.05 * (0.2 * 0.2)) = \boxed{0.0161}
\end{aligned}
$$