ECE2521: Analysis of Stochastic Processes

Homework 1: Topics 1 and 2 (Set Theory and Probability)

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Assigned Problems (10)

Textbook Problems (1.1 - 1.8): 1.6, 1.9, 2.1, 2.8, 2.14, 2.30, 2.34, and 2.119
Application Problems (1.9 - 1.10): Circuit tolerance, Dice Distribution

Textbook:

1.6. A random experiment consists of selecting two balls in succession from an urn containing two black balls and and one white ball.

(a) Specify the sample space for this experiment.

(b) Suppose that the experiment is modified so that the ball is immediately put back into the urn after the first selection. What is the sample space now?

(c) What is the relative frequency of the outcome (white, white) in a large number of repetitions of the experiment in part a? In part b?

(d) Does the outcome of the second draw from the urn depend in any way on the outcome of the first draw in either of these experiments?

1.9. The *sample mean* for a series of numerical outcomes $X(1), X(2), \ldots, X(n)$ of a sequence of random experiments is defined by

$$\langle X \rangle_n = \frac{1}{n}\sum_{j=1}^{n} X(j).$$

Show that the sample mean satisfies the recursion formula:

$$\langle X \rangle_n = \langle X \rangle_{n-1} + \frac{X(n) - \langle X \rangle_{n-1}}{n}, \qquad \langle X \rangle_0 = 0.$$

2.1. The (loose) minute hand in a clock is spun hard and the hour at which the hand comes to rest is noted.

(a) What is the sample space?

(b) Find the sets corresponding to the events: A = "hand is in first 4 hours"; B = "hand is between 2nd and 8th hours inclusive"; and D = "hand is in an odd hour."

(c) Find the events: $A \cap B \cap D$, $A^c \cap B$, $A \cup (B \cap D^c)$, $(A \cup B) \cap D^c$.

2.8. A number U is selected at random from the unit interval. Let the events A and B be: A = "U differs from 1/2 by more than 1/4" and B = "$1 - U$ is less than 1/2." Find the events $A \cap B$, $A^c \cap B$, $A \cup B$.

2.14. Let A, B, and C be events. Find expressions for the following events:

(a) Exactly one of the three events occurs.

(b) Exactly two of the events occur.

(c) One or more of the events occur.

(d) Two or more of the events occur.

(e) None of the events occur.

2.30. Use Corollary 7 to prove the following:

(a) $P[A \cup B \cup C] \le P[A] + P[B] + P[C]$.

(b) $P\left[\bigcup_{k=1}^{n} A_k\right] \le \sum_{k=1}^{n} P[A_k]$.

(c) $P\left[\bigcap_{k=1}^{n} A_k\right] \ge 1 - \sum_{k=1}^{n} P[A_k^c]$.

2.34. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$.

(a) Find the probabilities of A, B, $A \cap B$, and $A \cap C$.

(b) Find the probabilities of $A \cup B$, $A \cup C$, and $A \cup B \cup C$, first, by directly evaluating the sets and then their probabilities, and second, by using the appropriate axioms or corollaries.

2.119. Let A be any subset of S. Show that the class of sets $\{\varnothing, A, A^c, S\}$ is a field.

Application:

Practical 1. A circuit contains electrical components resistors, capacitors, and inductors. Each component is one of two types: low tolerance and high tolerance. The number of each component and type in the circuit is shown in the table:

	Resistors	Capacitors	Inductors
High tolerance	120	75	15
Low tolerance	80	25	10

We pick a component in the circuit, and we define the following events:

A = {component is a resistor}, B = {component is a capacitor}, C = {component is an inductor}, D = {high tolerance component}, and E = {low tolerance component}.

Find the probabilities of the following events: A, B, C, D, E, $A \cap D$, $C \cap E$, $A \cup B$, $(A \cup B) \cap D$. Explain what the events mean.

Practical 2. [MATLAB Problem] A pair of "fair" dice are rolled and the values of their up-faces are added to obtain an outcome of this random experiment.

(a) Determine the sample space for this experiment and the associated probabilities of its outcomes.

(b) Develop a MATLAB function [Ns,s] = Dice2Sum(N) that stores, in the array Ns, the number of times each sum-outcome has occurred given the number N of the repeated trials (N is a large number). The array s should contain the sum-outcomes for the corresponding count. Your function should

- simulate the rolling of a single die (i.e., generate a random integer between 1 and 6, inclusive),
- add two such outcomes to generate the sum-outcome,
- repeat this independently N times, and
- accumulate counts in Ns.

Using your function and N = 100000, compute the outcome probabilities via the relative frequency approach. Plot these probabilities using the bar-graph and verify your results in part (a) above. Submit printouts of the function, your script, and the plot.

(c) The following events are defined:
- A = {event of an odd number sum}
- B = {event of an integer multiple of 3 sum}

Determine Pr (A ∪ B) analytically and then verify it using your function. Submit a printout of your script.

Textbook Problems 1 through 8

1.1)

(a) Sample space S = {black-black, black-white, white-black}

(b) $S_{replace}$ = {black-black, black-white, white-black, white-white}

(c) Relative frequency $f_{ww,(a)} = 0$ and $f_{ww,(b)} = \frac{1}{9}$

(d) In (a) the second draw in dependent, but in (b) the second draw is independent due to replacement

1.2)

Prove:

$$begin \rightarrow \quad <X>_n = \frac{1}{n}\sum_{j=1}^{n} X(j) \tag{1}$$

$$bounds \rightarrow \quad = \frac{1}{n}\left(\sum_{j=1}^{n-1} X(j) + X(n)\right) \tag{2}$$

$$distribute \rightarrow \quad = \frac{1}{n}\sum_{j=1}^{n-1} X(j) + \frac{X(n)}{n} \tag{3}$$

$$rewrite \rightarrow \quad = \left(\frac{1}{n-1}\sum_{j=1}^{n-1} X(j) * \frac{n-1}{n}\right) + \frac{X(n)}{n} \tag{4}$$

$$definition \rightarrow \quad = \left(<X>_{n-1} * \frac{n-1}{n}\right) + \frac{X(n)}{n} \tag{5}$$

$$rewrite \rightarrow \quad = \left(<X>_{n-1} * \left(1 - \frac{1}{n}\right)\right) + \frac{X(n)}{n} \tag{6}$$

$$distribute \rightarrow \quad = <X>_{n-1} - \frac{<X>_{n-1}}{n} + \frac{X(n)}{n} \tag{7}$$

$$simplify...QED \rightarrow \quad = \boxed{<X>_{n-1} - \frac{X(n)- <X>_{n-1}}{n}} \tag{8}$$

1.3)

(a) Sample space S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

(b) Events:

$$A = \{1,2,3,4\}$$
$$B = \{2,3,4,5,6,7,8\}$$
$$D = \{1,3,5,7,9,11\}$$

(c) Events:

$$A \cap B \cap D = \{3\}$$
$$A^c \cap B = \{5,6,7,8\}$$
$$A \cup (B \cap D^c) = \{1,2,3,4,6,8\}$$
$$(A \cup B) \cap D^c = \{2,4,6,8\}$$

1.4)

Given information with bounds [0,1]:

$$
\begin{aligned}
A &:= |u-\frac{1}{2}| > \frac{1}{4} && \rightarrow && \frac{3}{4} < u < \frac{1}{4} \\
B &:= 1-u > \frac{1}{2} && \rightarrow && u > \frac{1}{2}
\end{aligned}
$$

Events:

$$
\begin{aligned}
A \cap B &= \{u \mid u > \frac{3}{4}\} \\
A^c \cap B &= \{u \mid \frac{1}{2} < u \leq \frac{3}{4}\} \\
A \cup B &= \{u \mid \frac{1}{4} > u > \frac{1}{2}\}
\end{aligned}
$$

1.5)

Expressions:

$$
\begin{aligned}
i) &\quad = 1 events && \rightarrow && (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \\
ii) &\quad = 2 events && \rightarrow && (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \\
iii) &\quad \geq 1 events && \rightarrow && A \cup B \cup C \\
iv) &\quad \geq 2 events && \rightarrow && (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C) \\
v) &\quad = 0 events && \rightarrow && A^c \cap B^c \cap C^c
\end{aligned}
$$

1.6)

Corollary 7: if A ⊂ B, then P[A] ≤ P[B] also implying that P[A∪B] ≤ P[A] + P[B]

(a) Prove that: $P[A \cup B \cup C] \quad \leq \quad P[A] + P[B] + P[C]$

$$
\begin{aligned}
commutativity &\quad \rightarrow \quad & P[A \cup B \cup C] &= P[(A \cup B) \cup C] && (1) \\
corollary &\quad \rightarrow \quad & &\leq P[A \cup B] + P[C] && (2) \\
corollary, QED &\quad \rightarrow \quad & &\leq \boxed{P[A] + P[B] + P[C]} && (3)
\end{aligned}
$$

(b) Prove that: $P[\cup_{k=1}^{n} A_k] \quad \leq \quad \sum_{k=1}^{n} P[A_k]$

$$
\begin{aligned}
bounds &\quad \rightarrow \quad & P[\cup_{k=1}^{n} A_k] &= P[\cup_{k=1}^{n-1} A_k] \cup P[A_n] && (1) \\
corollary &\quad \rightarrow \quad & &\leq P[\cup_{k=1}^{n-1} A_k] + P[A_n] && (2) \\
bounds &\quad \rightarrow \quad & &= \big(P[\cup_{k=1}^{n-2} A_k] \cup P[A_{n-1}]\big) + P[A_n] && (3) \\
corollary &\quad \rightarrow \quad & &\leq P[\cup_{k=1}^{n-2} A_k] + P[A_{n-1}] + P[A_n] && (4) \\
similarly ... QED &\quad \rightarrow \quad & &\leq \boxed{\sum_{k=1}^{n} P[A_k]} && (5)
\end{aligned}
$$

(c) Prove that: $P[\cap_{k=1}^{n} A_k] \quad \geq \quad 1 - \sum_{k=1}^{n} P[A_k^c]$

$$
\begin{aligned}
DeMorgans &\quad \rightarrow \quad & P[\cap_{k=1}^{n} A_k] &= 1 - P[\cup_{k=1}^{n} A_k^c] && (1) \\
result(b) ... QED &\quad \rightarrow \quad & &\geq \boxed{1 - \sum_{k=1}^{n} P[A_k^c]} && (2)
\end{aligned}
$$

1.7)

Given information with bonuds [-1,2]:

$$
\begin{array}{lclcl}
A := \{x < 0\} & \rightarrow & x < 0 & \rightarrow & \frac{1}{3} \\
B := \{|x - 0.5| < 0.5\} & \rightarrow & 0 < x < 1 & \rightarrow & \frac{1}{3} \\
C := \{x > 0.75\} & \rightarrow & x > 0.75 & \rightarrow & \frac{1.25}{3} = \frac{5}{12}
\end{array}
$$

<u>Events</u>:

(a)

$$
\begin{array}{ll}
P[A] = \frac{1}{3} & P[A \cap B] = \phi \\
P[B] = \frac{1}{3} & P[B \cap C] = \frac{1}{12} \\
P[C] = \frac{5}{12} & P[A \cap C] = \phi
\end{array}
$$

(b)

$$
\begin{array}{rclcll}
P[A \cup B] & = & \{[-1, 1)\} & or & P[A] + P[B] - P[A \cap B] & = \boxed{\frac{2}{3}} \\
P[A \cup C] & = & \{[-1, 0), (0.75, 2]\} & or & P[A] + P[C] - P[A \cap C] & = \boxed{\frac{3}{4}} \\
P[A \cup B \cup C] & = & \{[-1, 2]\} & or & P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[A \cap C] & = \boxed{1}
\end{array}
$$

1.8)

Using equation 2.49 (from page 74):

(i) (2.49a) $\rightarrow$ $\phi \in \mathcal{F}$

(ii) (2.49c) $\rightarrow$ $if\ A^c \in \mathcal{F} \quad then \quad A \in \mathcal{F}$

(iii) (2.49c) $\rightarrow$ $if\ A \in \mathcal{F} \quad then \quad A^c \in \mathcal{F}$

(iv) (2.49b) $\rightarrow$ $if\ A, A^c \in \mathcal{F} \quad then \quad A \cup A^c = S \in \mathcal{F}$

Application Problems 9 through 10

1.9)

	Resistor	Capacitor	Inductor	*total tol.*
High Tol.	120	75	15	*210*
Low Tol.	80	25	10	*115*
total comp.	*200*	*100*	*25*	***325***

Events: A{resistor}, B{capacitor}, C{inductor}, D{high tol.}, E{low tol.}. {A, B, C} independent from {D, E}. The chances of picking various components are:

i) $P[A] = \frac{200}{325} = \frac{8}{13}$ chance of resistor

ii) $P[B] = \frac{100}{325} = \frac{4}{13}$ chance of capacitor

iii) $P[C] = \frac{25}{325} = \frac{1}{13}$ chance of inductor

iv) $P[D] = \frac{210}{325} = \frac{42}{65}$ chance of high tol. component

v) $P[E] = \frac{115}{325} = \frac{23}{65}$ chance of low tol. component

vi) $P[A \cap D] = \frac{8}{13} * \frac{120}{200} = \frac{120}{325}$ chance of high tol. resistor

vii) $P[C \cap E] = \frac{1}{13} * \frac{10}{25} = \frac{10}{325}$ chance of low tol. inductor

viii) $P[A \cup B] = \frac{8}{13} + \frac{4}{13} = \frac{12}{13}$ chance of resistor or capacitor

ix) $P[A \cup B] \cap D = \frac{12}{13} * \frac{195}{300} = \frac{195}{325}$ chance of high tol. [resistor or capacitor]

1.10)

Context: Roll 2 dice 10^5 and observe distribution of sums:

(a)

$$Samplespace = \{2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 11, \quad 12\}$$
$$Probabilities = \{\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}\}$$

(b) Code, plot, and relative frequencies (outcome, count, relative frequency):

hw1_10_Dice2Sum.m

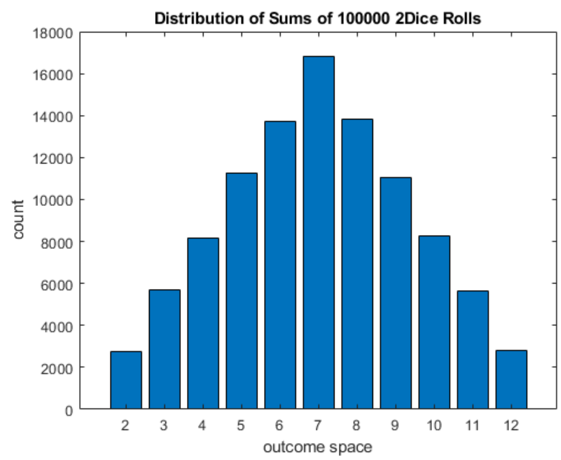
```
clear all;
close all;
clc;
format shortG

%hw 1.10, Seth So, 8/28/20
%context: simulate distribution for sum of [2] fair dice rolls
n = input("how many trials of 2 dice rolls? ");
[Ns,s] = Dice2Sum(n);

bar(s,Ns)
strTitle = sprintf('Distribution of Sums of %d 2Dice Rolls',n);
title(strTitle)
ylabel('count')
xlabel('outcome space')

relfreq = cat(1,s,Ns,Ns/n)

%function
function [Ns,s] = Dice2Sum(n)
Ns = zeros(1,11);
s = 2:12;
for i = 1:n
    sum = randperm(6,1) + randperm(6,1);
    Ns(sum-1) = Ns(sum-1) + 1;
    i = i+1;
end
end
```




```
how many trials of 2 dice rolls? 100000

relative_frequencies =

         2         3         4         5         6         7         8         9        10        11        12
      2748      5682      8167     11273     13717     16834     13802     11067      8264      5644      2802
   0.02748   0.05682   0.08167   0.11273   0.13717   0.16834   0.13802   0.11067   0.08264   0.05644   0.02802

>>
```

(c) A:={odd sum}, B:{multiple of 3 sum} find P[A ∪ B]:

$$A = \{3,5,7,9,11\} \quad , \quad B = \{3,6,9,12\}$$
$$A \cap B = \{3,9\} \quad , \quad A \cup B = \{3,5,6,7,9,11,12\}$$
$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \quad = \quad \frac{18}{36} + \frac{12}{36} - \frac{6}{36} \quad = \quad \boxed{\frac{24}{36} \approx 0.667}$$

Using the relative frequencies calculated in (b), this is verified with the simulation:

$$\begin{aligned} P[A \cup B] &= f_3 + f_5 + f_6 + f_7 + f_9 + f_{11} + f_{12} \\ &= 0.05682 + 0.1127 + 0.1372 + 0.1683 + 0.1107 + 0.05644 + 0.02802 \\ &= \boxed{0.670} \end{aligned}$$